sional space, and all our reasoning is in three-dimensional space: our line is length, breadth, and thickness, though the latter two may be infinitesimals; our surfaces are length, breadth, and thickness, though the last may be an infinitesimal. It is thus evident that we cannot pass from one order of space to another by any system of multiplication. Points cannot become a line; nor lines, a surface; nor surfaces, a solid.

It is true that a point in motion describes a line and must describe a line. A line may describe a line, if it follow along itself as a path; otherwise its motion must describe a surface. A surface (a plane and surface of a sphere only) may describe a surface if it follows along itself as a path; otherwise it must describe a solid. A solid in motion may describe an infinite number of solids, but its motion always describes a solid. It may be a solid of revolution, a prism, a tortuous prism or cylindroid etc. It cannot be conceived as describing space other than three-dimensional. We have no valid reason for assuming that there is any higher order of space in the ordinary signification than three-dimensional. Indeed there is the best of reasons for assuming that there is none.

Volumes have been written by psychologists and philosophers to explain the notion of space; and the more these philosophers write, the more they seem to think there is something mysterious about it, and the more they themselves become befogged in their reasoning about it. It is a mere matter of definition; everyone knows what it is and can usually define what he means by the word. One might as well attempt to prove that a straight line is straight, or that a circle is circular. Discussions of all self-evident truths are profitless; except it may be to show how such truths are apprehended, and it is extremely doubtful whether these succeed in the sense of giving mental satisfaction.

But what I wish most to protest against, is the use of fundamental untruths as the basis of rigorous reasoning, and insisting on the truth of the deductions when negatived by consciousness and common reason.

LANSING, MICH. CHAS. H. CHASE.

EDITORIAL COMMENT.

This communication on "Pseudo-Geometry" by Judge Charles H. Chase has a certain justification. It is an expression of common sense against being bulldozed into mysticism by the extravagances of a highly abstruse reasoning, and we endorse his protest so far

as to say that we too do not believe in pseudo-geometry. We have explained our own view on mathematics in a series of articles, published in *The Monist,* XIII, 273, 370, 493, and need not repeat ourselves here. In these articles we have not countenanced metageometry except as a method of space-construction which is possible, but which possesses little practical use because our old traditional geometry is simpler and better adapted to the solution of the problems of space-relations. Pangeometry as a higher generalization is helpful in order to understand certain problems because it makes Euclidean geometry appear as one special case among several possibilities.

The problem of the significance of metageometry as well as pangeometry is, however, not quite so simple as Judge Chase would have us believe. The reason of the mathematicians is in no way different from common sense. The mechanism of the two differs only as a precision machine would be different from any other one built in a rougher style. At the same time we must insist that a person operating the more delicate micrometer is just as apt to make mistakes as the man who uses a foot-rule. Mathematicians and philosophers have made mistakes as well as the people of common sense, and so we must not be astonished to find in the realm of abstruse thought statements which common sense would deem extravagant or even obviously self-contradictory.

We must bear in mind that geometry together with arithmetic and logic are constructions of pure thought. They are not realities, nor do they represent real things. They represent mere relations, and we are at liberty to make these constructions according to the principles which we lay down at the start. The simplest principle will lead to the simplest system, and it is obvious that the simplest system will be most useful for dealing with actual things. Our geometrical constructions are like models representing relations in pure thought which may exist in reality.

We must distinguish therefore between geometrical space and real space. It is a pity that both are denoted by the same word. Actual space is an objective feature of the things with which we become acquainted in our experience. Our space-conception, however, is mere thought, but it is useful in calculating the space-relations of real things.

It is possible to construct several spaces in the realm of pure thought, but it is not possible to assume that there are several spaces in the objective world. The space of objective existence is simply

the juxtaposition of things, or the scope of the changes that may be made in this juxtaposition. In other words, space is the scope of motion. The two definitions are practically the same, but I would prefer the latter one as more descriptive of both the nature of space and the origin of our space-conception. The former definition, "the juxtaposition of things" is static; the latter, "the scope of motion," is dynamic, and since life is action, the latter seems more appropriate.

By moving about we construct our space-conception from the fact of our own motility, which means that we can move about and change the place of our own position as well as the position of things in our surroundings.

According to Kant our space-conception is *a priori,* which means in Kantian terminology that it can be constructed without resorting to sense-impressions. This is true, but nevertheless geometry is not absolutely *a priori.* It makes use of the idea of motion which we tacitly retain and utilize in constructing our field of action, i. e., our general conception of space.

The geometrician experiments with his motion, and constructs points, lines, surfaces and solids. None of them are real, but all of them are called true if they are constructed with rigid consistency according to the principle laid down. A theorem is true if it agrees with the necessary result acomplished by construction—a result which in most cases is uniquely determined.

There are cases in which the function can not be executed. Such is especially the extraction of the square root of minus quantities, and so mathematics is nonplused when it is confronted with the solution of $\sqrt{-1}$. This is called the irrational, but it had better be called "the unrealizable." It is not against reason; the simple truth is, it can not be done.

We agree perfectly with Judge Chase when he denounces mathematicians for speaking of the sum of the angles of a plane triangle as being more or less than two right angles. If a triangle is plane the sum of the angles is unequivocally equal to two right angles, for it has been made so by construction, and there can be no quibbling about it. If there were any doubt we would have to insist that the triangle, the sum of whose angles is more or less than 180°, can not be a plane triangle but must belong to another system of geometry. When we deal with real space we may boldly say that the method of measuring and determining points by three co-ordinates will always prove sufficient, and this is all that can reasonably be meant by the statement that space, viz., the objective space of things, is

tridimensional, but if we deal with space-conceptions as mathematical constructions we may very well build up systems of different manifoldness, and need not limit ourselves to three dimensions. By moving a point we create a line; by moving a line in another but its own direction we create a surface; by moving a surface not in its own plane but in some other direction we create a solid. Now if we move a real solid in actual space we produce a change of place but we do not create a new dimension; but if we move a mathematical solid at the same time retaining its path just as we did before when creating the surface from a line and the solid from a surface we shall not leave this geometrical solid unaltered but we shall produce a new kind of a body which is four-dimensional, and which augments the significance of the solid just as the path of the surface increased the significance of the plane.

The difficulty consists solely in the fact that actual space, viz., our scope of motion, allows us to go in any direction possible, and however much we may strain our imagination, we can not find a direction not already contained in the solid. So in reality we can not fulfil the condition that in constructing the next higher figure we must move at a right angle in a direction not contained in the figure with which we start. But what is impossible in actual space is allowable to the imagination.

We must grant that the four-dimensional figure is imaginary, but it is no more imaginary than mathematical lines, surfaces and solids. The space through which the solid travels is, as it were, (if we cling to the motion of tridimensional space) in part covered twice, and it is true we can have no conception of its appearance.

The realm of thought is wide, and so nothing will prevent us from making any imaginary construction of four-dimensional bodies, and the strange thing about it is that though we can not picture it as a sense-perceptible form we can determine the laws of four-dimensional bodies with absolute exactness. Take for instance a line of three linear inches. If we move it at right angles with its own direction the surface will contain nine square inches. If we now move the surface again in a direction at right angles to the plane we will have a solid of twenty-seven cubic inches, and here our mode of representing further results of motion in a sense-perceptible image stops. But suppose we should move it again at the right angle of its own former construction (whatever that may be) we shall construct a figure, unrealizable though it may be, which will consist of 81 inches, each one being possessed of four directions.

This method of geometrizing is, as Judge Chase rightly says, a kind of mental gymnastics, but in principle it is in no wise different from Euclidean geometry and solid geometry. It is a construction of pure thought and is the product of generalizing the idea of dimension which creates new possibilities, incompatible though they may be with actual space.

There are some who talk about the possibility that our objective space may not be Euclidean but like one of the hyperspaces of modern mathematicians. But the hypothesis is worthless if we consider that objective space does not consist of objective things, but is a mere scope of motion. Whatever may be said in justification and in praise of metageometry, one thing is sure: There is no mathematician who for the sake of calculating distances, loci, or angles in the conditions of actual space would utilize or seriously recommend the use of any other but the Euclidean geometry.

SPACE OF FOUR DIMENSIONS.

The several conceptions of space of more than three dimensions are of a purely abstract nature, yet they are by no means vague, but definitely determined by the conditions of their construction. Therefore we can determine their abstract thought and very details with perfect exactness and formulate in abstract thought the laws of four-, five-, six-, and n-dimensional space. The difficulty with which we are beset in constructing n-dimensional spaces consists in our inability to make them representable to our senses. Here we are confronted with what may be called the limitations of our mental constitution. These limitations, if such they be, are conditioned by the nature of our mode of motion, which if reduced to a mathematical system needs for a description in definite terms three co-ordinates, and this means that our space-conception is tridimensional.

We ourselves are tridimensional; we can measure the space in which we move with three co-ordinates, yet we can definitely say that if space were four-dimensional, a body constructed of two factors, so as to have a four-dimensional solidity, would be expressed in the formula:

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

We can calculate, compute, excogitate, and describe all the characteristics of four-dimensional space, so long as we remain in the realm of abstract thought and do not venture to make use of our motility and execute our plan in an actualized construction of